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ANALYTICAL, THREE-DIMENSIONAL ELASTICITY SOLUTIONS TO SOME PLATE PROBLEMS, AND SOME OBSERVATIONS ON MINDLIN'S PLATE THEORY

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Abstract—Solutions are obtained, using the exact three-dimensional theory of elasticity, to (i) the eigenvalue problem of buckling under biaxial compression or the free lateral vibration of a simply supported rectangular plate with orthotropic stress–strain properties, and (ii) the static response of the same plate to a lateral load that varies sinusoidally in two directions. The eigenvalue in problem (i) and the lateral deflections of both the surface and middle plane of the plate, as well as the bending strains, in problem (ii) are obtained in the form of series expansions in even powers of the plate thickness. Exact algebraic expressions are presented for the first two coefficients in the case of orthotropic plates; additional coefficients can be obtained if required, and are given for the much simpler case of isotropic plates. In all cases the first term agrees with classical plate theory. The solutions are compared with those obtained from Mindlin's plate theory. In neither problem is it found to be possible, in general, to choose values for Mindlin's effective shear moduli to make the Mindlin solution agree with the first two terms of the exact solution. There are, however, two exceptions to this, namely a restricted class of orthotropic materials, embracing all isotropic ones, in which the elastic constants satisfy a certain condition, and the case of cylindrical bending when the Mindlin plate reduces to a Timoshenko beam of wide rectangular cross-section. In both these exceptional cases appropriate values for the effective shear moduli are obtained.

1. INTRODUCTION

Classical plate theory (CPT) assumes that (i) the lateral displacement w is constant through the thickness, (ii) normals to the middle surface remain both straight and normal after deformation, and usually (iii) in the dynamic case the rotational inertia, arising from u, v displacements parallel to the middle surface due to bending, is negligible. With the possible exception of edge effects, the error in CPT is $O(h^2/\lambda^2)$ where h is the thickness and λ a typical "half-wavelength", or characteristic length, of the bent surface. For isotropic plates the error is usually very small in practice, though it can be significant at the shorter wavelengths associated with high frequency vibration modes. Nevertheless there has been much attention paid to the development of higher order plate theories with the implicit aim of reducing the error to $O(h^4/\lambda^4)$ or less. Notable among these are the theories of Reissner[1] and Mindlin[2]. There have also been other, more recent, theories proposed but they will not be mentioned further here.

In recent years, however, the development of fibre-reinforced composites has resulted in greater interest being shown in higher order theories. The reason is that in such plates the shear modulus associated with transverse (i.e. "through-the-thickness") shearing stresses is often very small compared with the elastic moduli associated with the bending stresses; this results in CPT becoming inadequate at much smaller h/λ ratios than in the isotropic case.

In order to assess the accuracy of approximate higher order theories it is obviously advantageous if some "exact" solutions of plate problems, based upon three-dimensional elasticity theory, are available for comparison. Such solutions have been derived for certain problems associated with the buckling, vibration or static loading of simply supported, orthotropic rectangular plates, of both homogeneous and laminated types, notably by Srinivas and co-workers[3–6] and Pagano[7]. They obtained many *numerical* solutions in this way and compared them with various approximate theories.

In this paper we shall obtain *algebraic* solutions for the same set of problems considered by Srinivas and co-workers except that, in order to keep the algebra within manageable

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bounds, we shall confine ourselves to *homogeneous* plates only. The problems considered are (i) the eigenvalue problem arising from either the buckling under biaxial compression or the free vibration of the plate, and (ii) the response to a static, lateral loading with intensity varying sinusoidally in both the x- and y-directions. Both problems result in a doubly sinusoidal mode of deformation, enabling a solution to be obtained by separation of the variables.

We shall start with the basic equations of three-dimensional, orthotropic elasticity, but whereas Srinivas and co-workers[3–6] resorted to computation at an early stage and obtained numerical solutions for specific cases our solutions will be entirely algebraic. The eigenvalue in problem (i) and the deflections and bending strains in problem (ii) are in the form of series expansions in powers of h^2. Specific expressions are given for the first two terms in the case of a general orthotropic material; more terms are obtainable but they become increasingly complicated algebraically. It is, however, rather easy to obtain additional terms of the series for an isotropic plate, and this has been done.

These solutions are then used to discuss the accuracy of Mindlin's plate theory. What appear to be new results are obtained for the effective shear modulus in the case of isotropic plates, enabling the error to be reduced from $O(h^2/\lambda^2)$ to $O(h^4/\lambda^4)$, and smaller still for all practical purposes in the eigenvalue problem. But for orthotropic plates it is shown that in general it is impossible to select values for Mindlin's two effective shear moduli which reduce the error to $O(h^4/\lambda^4)$ for all values of the ratio between the wavelengths in the x- and y-directions. The only exception to this is a class of orthotropic materials, embracing all isotropic ones, in which the elastic constants satisfy a very restrictive condition. It is possible also in the case of cylindrical bending of an orthotropic plate, in which case the Mindlin plate behaves effectively as a Timoshenko beam of *wide* rectangular cross-section.

2. THREE-DIMENSIONAL THEORY OF ELASTICITY SOLUTIONS

2.1. *The basic equations of three-dimensional elasticity*

Consider a homogeneous elastic plate, of uniform thickness h, subjected to uniform static *compressive* stresses σ_x^0 and σ_y^0 parallel to axes x and y lying in the middle plane of the plate. Suppose that small displacements $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ occur from this datum state of uniform biaxial compression. Then, if there are no body forces, the equations of motion of an element in the x-, y- and z-directions are

$$\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}&=\mathscr{L}(u)\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}&=\mathscr{L}(v)\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}&=\mathscr{L}(w)\end{aligned}\tag{1}$$

where the differential operator $\mathscr{L}$ is defined by

$$\mathscr{L}=\sigma_x^0\frac{\partial^2}{\partial x^2}+\sigma_y^0\frac{\partial^2}{\partial y^2}+\rho\frac{\partial^2}{\partial t^2}\tag{2}$$

and ρ is the density. The stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz} are additional to those in the datum state.

The operator $\mathscr{L}$ which appears on the right-hand sides of eqns (1) takes account not only of the inertia forces but also of the destabilizing effect of σ_x^0 and σ_y^0 in all three directions. The simplest way of deriving these terms is via a variational principle, using Green's strain tensor to relate the linear strains in the x- and y-directions to the displacements u, v, w by the quadratic expressions $(\partial u/\partial x) + \varepsilon_x'$ and $(\partial v/\partial y) + \varepsilon_y'$, respectively, where

$$\varepsilon_x' = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \qquad \varepsilon_y' = \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]. \tag{3}$$

The constitutive equations for an orthotropic material with principal axes parallel to x, y, z are

$$\begin{aligned} \sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z, & \tau_{yz} &= c_{44}\gamma_{yz} \\ \sigma_y &= c_{12}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z, & \tau_{xz} &= c_{55}\gamma_{xz} \\ \sigma_z &= c_{13}\varepsilon_x + c_{23}\varepsilon_y + c_{33}\varepsilon_z, & \tau_{xy} &= c_{66}\gamma_{xy} \end{aligned} \tag{4}$$

where the strains are given by

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{aligned} \tag{5}$$

Substitution of eqns (4) and (5) into eqns (1) gives three equations of motion expressed in terms of the displacements, as follows:

$$\begin{aligned} &\left(c_{11}\frac{\partial^2}{\partial x^2} + c_{66}\frac{\partial^2}{\partial y^2} + c_{55}\frac{\partial^2}{\partial z^2} - \mathscr{L}\right)u + (c_{12} + c_{66})\frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{55})\frac{\partial^2 w}{\partial x \partial z} = 0 \\ &(c_{12} + c_{66})\frac{\partial^2 u}{\partial x \partial y} + \left(c_{66}\frac{\partial^2}{\partial x^2} + c_{22}\frac{\partial^2}{\partial y^2} + c_{44}\frac{\partial^2}{\partial z^2} - \mathscr{L}\right)v + (c_{23} + c_{44})\frac{\partial^2 w}{\partial y \partial z} = 0 \\ &(c_{13} + c_{55})\frac{\partial^2 u}{\partial x \partial z} + (c_{23} + c_{44})\frac{\partial^2 v}{\partial y \partial z} + \left(c_{55}\frac{\partial^2}{\partial x^2} + c_{44}\frac{\partial^2}{\partial y^2} + c_{33}\frac{\partial^2}{\partial z^2} - \mathscr{L}\right)w = 0. \end{aligned} \tag{6}$$

In the special case of an isotropic material we have

$$\begin{aligned} c_{11} = c_{22} = c_{33} &= \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} = \frac{2(1 - \nu)G}{(1 - 2\nu)} \\ c_{12} = c_{13} = c_{23} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} = \frac{2\nu G}{(1 - 2\nu)} \end{aligned} \tag{7}$$

and

$$c_{44} = c_{55} = c_{66} = \frac{E}{2(1 + \nu)} = G$$

where E, G and ν are the Young's modulus, shear modulus and Poisson's ratio, respectively. Equations (6) then take the concise form

$$\left\{\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right\} e + (1 - 2\nu)\mathscr{H}\{u; v; w\} = 0 \tag{8}$$

where e is the dilatation, defined by

$$e = (\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) \tag{9}$$

and the differential operator $\mathscr{H}$ is defined by

$$\mathscr{H} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{G}\left(\sigma_x^0 \frac{\partial^2}{\partial x^2} + \sigma_y^0 \frac{\partial^2}{\partial y^2} + \rho \frac{\partial^2}{\partial t^2}\right). \tag{10}$$

2.2. *The general solution for a doubly sinusoidal mode of displacement*

The problems for which we shall obtain solutions are all concerned with a mode of displacement of the form

$$\begin{aligned} u(x, y, z, t) &= U(z) \cos \alpha x \sin \beta y \cos \omega t \\ v(x, y, z, t) &= V(z) \sin \alpha x \cos \beta y \cos \omega t \\ w(x, y, z, t) &= W(z) \sin \alpha x \sin \beta y \cos \omega t. \end{aligned} \tag{11}$$

Using eqns (4) and (5) it can then be seen that $w = 0$, $v = 0$ and $\sigma_x = 0$ on the edges $x = 0$ and a, whilst $w = 0$, $u = 0$ and $\sigma_y = 0$ on the edges $y = 0$ and b of a rectangular plate, provided that α and β are such that $\alpha a/\pi$ and $\beta b/\pi$ are integers. For convenience we shall refer to edges with these boundary conditions as *simply supported.* Clearly the conventional definition of a simply supported edge in classical thin plate theory conforms to this more general definition.

If eqns (11) are substituted into eqns (6) we obtain the following simultaneous differential equations for U, V and W

$$\begin{bmatrix} (\mathscr{D}^2 - a_5) & -(c_{44}/c_{55})^{1/2} b_3 & (c_{33}/c_{55})^{1/2} b_4 \mathscr{D} \\ -(c_{55}/c_{44})^{1/2} b_3 & (\mathscr{D}^2 - a_4) & (c_{33}/c_{44})^{1/2} b_5 \mathscr{D} \\ -(c_{55}/c_{33})^{1/2} b_4 \mathscr{D} & -(c_{44}/c_{33})^{1/2} b_5 \mathscr{D} & (\mathscr{D}^2 - a_3) \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{O} \tag{12}$$

where $\mathscr{D}$ is the operator d/dz

$$\begin{aligned} a_3 &= (c_{55}\alpha^2 + c_{44}\beta^2 - \xi)/c_{33}, & b_3 &= (c_{12} + c_{66})\alpha\beta/(c_{44}c_{55})^{1/2} \\ a_4 &= (c_{66}\alpha^2 + c_{22}\beta^2 - \xi)/c_{44}, & b_4 &= (c_{13} + c_{55})\alpha/(c_{33}c_{55})^{1/2} \\ a_5 &= (c_{11}\alpha^2 + c_{66}\beta^2 - \xi)/c_{55}, & b_5 &= (c_{23} + c_{44})\beta/(c_{33}c_{44})^{1/2} \end{aligned} \tag{13}$$

and ξ is defined by

$$\xi = \alpha^2 \sigma_x^0 + \beta^2 \sigma_y^0 + \rho\omega^2. \tag{14}$$

If U and V are eliminated from eqn (12) we obtain a sixth-order differential equation for $W(z)$, namely

$$(\mathscr{D}^6 - Q_1 \mathscr{D}^4 + Q_2 \mathscr{D}^2 - Q_3) W = 0 \tag{15}$$

where

$$\begin{aligned} Q_1 &= a_3 + a_4 + a_5 - b_4^2 - b_5^2 \\ Q_2 &= a_3 a_4 + a_4 a_5 + a_5 a_3 + 2 b_3 b_4 b_5 - b_3^2 - a_4 b_4^2 - a_5 b_5^2 \\ Q_3 &= a_3(a_4 a_5 - b_3^2). \end{aligned} \tag{16}$$

Let the roots of the auxiliary equation of eqn (15) be $\pm q_1$, $\pm q_2$ and $\pm q_3$. Then

$$
\begin{aligned}
Q_1 &= q_1^2 + q_2^2 + q_3^2 \\
Q_2 &= q_1^2 q_2^2 + q_2^2 q_3^2 + q_3^2 q_1^2 \\
Q_3 &= q_1^2 q_2^2 q_3^2.
\end{aligned}
\tag{17}
$$

In the analysis that follows it is fortunately never necessary to obtain expressions for the individual roots, q_1, q_2 and q_3. Only eqns (17), and the values of certain other combinations of the three roots that can be deduced from them, are required, and it is immaterial whether the individual roots are real, imaginary or complex. The other combinations that appear in the analysis are always reducible to the form $I_{m,n}$ or $J_{m,n}$, where

$$I_{m,n} = \sum{}^* [q_k^{2m}(q_i^{2n} - q_j^{2n})] \tag{18}$$

$$J_{m,n} = \sum{}^* [q_i^{2m} q_j^{2m}(q_i^{2n} - q_j^{2n})] \tag{19}$$

and m and n are positive integers. The symbol $\sum^*$ denotes a summation extending over three terms, formed by allocating the values (1, 2, 3), (2, 3, 1) and (3, 1, 2) to the suffixes (i, j, k) in the general term shown.

Certain relations between the various $I_{m,n}$ and $J_{m,n}$ quantities are given in Appendix A, and in particular it is shown that they can all be expressed in the form $J_{1,1}$ times a polynomial function of Q_1, Q_2 and Q_3.

The general solutions of the differential equations, eqns (12), are

$$
\begin{aligned}
U(z) &= -\sum_{i=1}^{3} q_i e_i (K_i \sinh q_i z + K_i' \cosh q_i z) \\
V(z) &= -\sum_{i=1}^{3} q_i f_i (K_i \sinh q_i z + K_i' \cosh q_i z) \\
W(z) &= \sum_{i=1}^{3} (K_i \cosh q_i z + K_i' \sin q_i z)
\end{aligned}
\tag{20}
$$

where the K_i and K_i' are constants of integration, and

$$
\begin{aligned}
e_i &= (c_{33}/c_{55})^{1/2}[b_3 b_5 + b_4(q_1^2 - a_4)]/Z_i \\
f_i &= (c_{33}/c_{44})^{1/2}[b_3 b_4 + b_5(q_i^2 - a_5)]/Z_i
\end{aligned}
\tag{21}
$$

in which

$$Z_i = (q_i^2 - a_4)(q_i^2 - a_5) - b_3^2. \tag{22}$$

[Note that $U = -q_i e_i$, $V = -q_i f_i$ is the solution of the first two of eqns (12) with $\mathscr{D}$ replaced by q_i and $W = 1$.]

The stresses acting on planes perpendicular to the z-axis, which are required later, are obtainable from eqns (4), (5), (11) and (20). They are given by

$$
\begin{aligned}
\sigma_z &= \hat{\sigma}_z \sin \alpha x \sin \beta y \cos \omega t \\
\tau_{xz} &= \hat{\tau}_{xz} \cos \alpha x \sin \beta y \cos \omega t \\
\tau_{yz} &= \hat{\tau}_{yz} \sin \alpha x \cos \beta y \cos \omega t
\end{aligned}
\tag{23}
$$

where the quantities bearing a circumflex are functions of z given by

$$\begin{aligned}
\hat{\sigma}_z &= \sum_{i=1}^{3} q_i(c_{13}\alpha e_i + c_{23}\beta f_i + c_{33})(K_i \sinh q_i z + K_i' \cosh q_i z) \\
\hat{\tau}_{xz} &= c_{55}\sum_{i=1}^{3} (\alpha - q_i^2 e_i)(K_i \cosh q_i z + K_i' \sinh q_i z) \\
\hat{\tau}_{yz} &= c_{44}\sum_{i=1}^{3} (\beta - q_i^2 f_i)(K_i \cosh q_i z + K_i' \sinh q_i z).
\end{aligned} \tag{24}$$

2.3. *The eigenvalue problem and its solution*

We shall now solve, simultaneously, the problems of:

(a) initial buckling of a simply supported rectangular plate under the uniform biaxial, static, compressive stresses σ_x^0 and σ_y^0, and

(b) free lateral vibration at a frequency ω of a simply supported rectangular plate about the datum state of uniform biaxial, static, compression σ_x^0 and σ_y^0.

In both cases the mode is of the form of eqns (11). In the buckling problem we have $\omega = 0$, and ξ, defined in eqn (14), is then a load parameter. In the vibration problem we postulate that σ_x^0 and σ_y^0 are specified but are not large enough to cause the plate to buckle, and ξ is then a frequency parameter. Both are eigenvalue problems, and since σ_x^0, σ_y^0 and ω appear only in combination in the form of the parameter ξ, its eigenvalues simultaneously give the buckling loads in problem (a), and the natural frequencies in problem (b). Moreover, the buckling and vibration modes are identical.

It is clear that the lowest buckling load and the lowest natural frequency, for given values of the wavelength parameters α and β, will be associated with modes in which $W(z)$ is an even-valued function whilst $U(z)$ and $V(z)$ are odd-valued functions. Hence, referring to eqns (20), we may assume that $K_i' = 0$ $(i = 1, 2, 3)$. We now impose the condition that the plate surfaces, $z = \pm\frac{1}{2}h$, are free from traction, so that

$$\hat{\sigma}_z = \hat{\tau}_{xz} = \hat{\tau}_{yz} = 0 \quad \text{at } z = \pm\tfrac{1}{2}h.$$

Equations (24) then give

$$\begin{bmatrix} q_1^2F_1T_1 & q_2^2F_2T_2 & q_3^2F_3T_3 \\ (\alpha - e_1q_1^2) & (\alpha - e_2q_2^2) & (\alpha - e_3q_3^2) \\ (\beta - f_1q_1^2) & (\beta - f_2q_2^2) & (\beta - f_3q_3^2) \end{bmatrix} \begin{bmatrix} K_1\cosh\frac{1}{2}hq_1 \\ K_2\cosh\frac{1}{2}hq_2 \\ K_3\cosh\frac{1}{2}hq_3 \end{bmatrix} = \mathbf{O} \tag{25}$$

where

$$F_i = c_{13}\alpha e_i + c_{23}\beta f_i + c_{33} \tag{26}$$

and

$$T_i = \tanh(\tfrac{1}{2}hq_i)/(\tfrac{1}{2}hq_i).$$

Alternatively T_i can be expanded into the form of a series

$$T_i = 1 - \left(\frac{h^2}{12}\right)q_i^2 + \frac{6}{5}\left(\frac{h^2}{12}\right)^2 q_i^4 - \frac{51}{35}\left(\frac{h^2}{12}\right)^3 q_i^6 + \frac{62}{35}\left(\frac{h^2}{12}\right)^4 q_i^8 - \frac{4146}{1925}\left(\frac{h^2}{12}\right)^5 q_i^{10} + \cdots \tag{27}$$

For a non-trivial solution of eqn (25) to exist, the determinant of the matrix must vanish and this gives the following characteristic equation

$$\sum{}^{*} [q_i^2 F_i T_i\{\alpha(q_j^2 f_j - q_k^2 f_k) + \beta(q_k^2 e_k - q_j^2 e_j) + q_j^2 q_k^2(e_j f_k - e_k f_j)\}] = 0 \tag{28}$$

where the summation is as defined in the sentence following eqn (19).

If eqns (26) and (27) are substituted into eqn (28) the resulting equation can be expressed as

$$\psi^{(0)} - \left(\frac{h^2}{12}\right)\psi^{(1)} + \frac{6}{5}\left(\frac{h^2}{12}\right)^2\psi^{(2)} - \frac{51}{35}\left(\frac{h^2}{12}\right)^3\psi^{(3)} + \cdots = 0 \tag{29}$$

where

$$\begin{aligned}\psi^{(r)} = \sum{}^{*}[q_i^{2r+2}(c_{13}\alpha e_i + c_{23}\beta f_i + c_{33})\{\alpha(q_j^2 f_j - q_k^2 f_k) + \beta(q_k^2 e_k - q_j^2 e_j) \\ + q_j^2 q_k^2(e_j f_k - e_k f_j)\}].\end{aligned} \tag{30}$$

Note that the coefficient of $\psi^{(r)}$ in eqn (29) is the same as that of q_i^{2r} in the series expansion of T_i in eqn (27). After a cyclic modification of the suffixes i, j, k on some of the terms in eqns (30) it can be written as

$$\begin{aligned}\psi^{(r)} = c_{13}\alpha\{\alpha H_{11}^{(r)} + \beta H_{12}^{(r)} + H_{13}^{(r)}\} + c_{23}\beta\{\alpha H_{21}^{(r)} + \beta H_{22}^{(r)} + H_{23}^{(r)}\} \\ + c_{33}\{\alpha H_{31}^{(r)} + \beta H_{32}^{(r)} + H_{33}^{(r)}\}\end{aligned} \tag{31}$$

where

$$\begin{aligned}
H_{11}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_i^{2r} e_i f_j - q_j^{2r} e_j f_i) \\
H_{12}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_j^{2r} - q_i^{2r}) e_i e_j \\
H_{13}^{(r)} &= Q_3 \sum{}^{*} q_k^{2r} e_k(e_i f_j - e_j f_i) \\
H_{21}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_i^{2r} - q_j^{2r}) f_i f_j \\
H_{22}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_j^{2r} e_i f_j - q_i^{2r} e_j f_i) \\
H_{23}^{(r)} &= Q_3 \sum{}^{*} q_k^{2r} f_k(e_i f_j - e_j f_i) \\
H_{31}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_i^{2r} f_j - q_j^{2r} f_i) \\
H_{32}^{(r)} &= \sum{}^{*} q_i^2 q_j^2(q_j^{2r} e_i - q_i^{2r} e_j) \\
H_{33}^{(r)} &= Q_3 \sum{}^{*} q_k^{2r}(e_i f_j - e_j f_i).
\end{aligned} \tag{32}$$

By using eqns (16), (17), (21) and (22) all of the $H_{\mu\nu}^{(r)}$ summations defined in eqns (32) can be expressed in terms of the $I_{m,n}$ or $J_{m,n}$ types, as defined in eqns (18) and (19). These in turn can all be expressed in terms of $J_{1,1}$ as explained in Appendix A, and this enables the $H_{\mu\nu}^{(r)}$ summations to be evaluated. It turns out that they all contain a common factor, κ, defined by

$$\kappa = (c_{33}/Z_1 Z_2 Z_3)(c_{44}c_{55})^{-1/2}(b_3^2 - a_4 a_5)[b_4 b_5(a_4 - a_5) + b_3(b_4^2 - b_5^2)]J_{1,1} \tag{33}$$

and values of $\kappa^{-1}H_{\mu\nu}^{(r)}$ for $r = 0, 1$ and 2 are listed in Table 1.

Table 1. Values of the $H_{\mu\nu}^{(r)}$ summations, defined by eqns (32), for $r = 0$, 1 and 2

r	0	1	2
$\kappa^{-1}H_{11}^{(r)}$	-1	$(b_4^2 - a_5)$	$Q_1(b_4^2 - a_5) + Q_2 - a_3a_4$
$\kappa^{-1}H_{22}^{(r)}$	-1	$(b_5^2 - a_4)$	$Q_1(b_5^2 - a_4) + Q_2 - a_3a_5$
$(c_{55}/c_{44})^{1/2}\kappa^{-1}H_{12}^{(r)}$	0	$(b_4b_5 - b_3)$	$Q_1(b_4b_5 - b_3) + a_3b_3$
$(c_{44}/c_{55})^{1/2}\kappa^{-1}H_{21}^{(r)}$	0	$(b_4b_5 - b_3)$	$Q_1(b_4b_5 - b_3) + a_3b_3$
$(c_{55}/c_{33})^{1/2}\kappa^{-1}H_{13}^{(r)}$	0	$-a_3b_4$	$-a_3(Q_1b_4 + b_3b_5 - a_4b_4)$
$(c_{44}/c_{33})^{1/2}\kappa^{-1}H_{23}^{(r)}$	0	$-a_3b_5$	$-a_3(Q_1b_5 + b_3b_4 - a_5b_5)$
$(c_{33}/c_{55})^{1/2}\kappa^{-1}H_{31}^{(r)}$	b_4	$[b_4(Q_1 - a_4) + b_3b_5]$	$Q_1[b_4(Q_1 - a_4) + b_3b_5] - b_4Q_2$
$(c_{33}/c_{44})^{1/2}\kappa^{-1}H_{32}^{(r)}$	b_5	$[b_5(Q_1 - a_5) + b_3b_4]$	$Q_1[b_5(Q_1 - a_5) + b_3b_4] - b_5Q_2$
$\kappa^{-1}H_{33}^{(r)}$	$-a_3$	$-a_3(a_3 - b_4^2 - b_5^2)$	$-Q_1a_3(a_3 - b_4^2 - b_5^2) + a_3Q_2 - Q_3$

It is now convenient to define $\Psi^{(r)}$ by

$$\Psi^{(r)} = \kappa^{-1}\psi^{(r)}. \tag{34}$$

Expressions for $\Psi^{(0)}$, $\Psi^{(1)}$ and $\Psi^{(2)}$ are readily obtainable using Table 1 and eqns (13), (16) and (31); they are as follows

$$\Psi^{(0)} = \xi \tag{35}$$

$$\Psi^{(1)} = [\bar{c}_{11}\alpha^4 + 2(\bar{c}_{12} + 2c_{66})\alpha^2\beta^2 + \bar{c}_{22}\beta^4] - \xi\left[\alpha^2 + \beta^2 + \frac{2(c_{13}\alpha^2 + c_{23}\beta^2)}{c_{33}}\right] - \frac{\xi^2}{c_{33}} \tag{36}$$

$$\begin{aligned}\Psi^{(2)} = {} & \left[\left(\frac{\bar{c}_{11}}{c_{55}} - \frac{2c_{13}}{c_{33}}\right)\alpha^2 + \left(\frac{\bar{c}_{22}}{c_{44}} - \frac{2c_{23}}{c_{33}}\right)\beta^2 - \xi\left(\frac{1}{c_{33}} + \frac{1}{c_{44}} + \frac{1}{c_{55}} - \frac{c_{66}}{c_{44}c_{55}}\right)\right]\Psi^{(1)} \\ & + \left(\frac{\alpha^2}{c_{44}} + \frac{\beta^2}{c_{55}} - \frac{\xi}{c_{44}c_{55}}\right)\left[\alpha^2\beta^2\{(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\} + \xi\left\{\left(\bar{c}_{11} - \frac{2c_{13}c_{66}}{c_{33}}\right)\alpha^2\right.\right. \\ & \left.\left. + \left(\bar{c}_{22} - \frac{2c_{23}c_{66}}{c_{33}}\right)\beta^2\right\} - \xi^2\left(1 + \frac{c_{66}}{c_{33}}\right)\right] - \xi Q_2\end{aligned} \tag{37}$$

where

$$\bar{c}_{11} = c_{11} - (c_{13}^2/c_{33}), \qquad \bar{c}_{22} = c_{22} - (c_{23}^2/c_{33}), \qquad \bar{c}_{12} = c_{12} - (c_{13}c_{23}/c_{33}). \tag{38}$$

By putting $\sigma_z = 0$ in eqns (4) and eliminating the corresponding value of ε_z from the equations for σ_x and σ_y it can be seen that $\bar{c}_{11}$, $\bar{c}_{22}$ and $\bar{c}_{12}$ are modified elastic constants associated with a state of plane stress (see eqns (68)).

It can be seen from eqn (15) that

$$q_i^2 = Q_1 - Q_2q_i^{-2} + Q_3q_i^{-4}.$$

Equations (32) and (31) respectively then show that

$$H_{\mu\nu}^{(r+1)} = Q_1H_{\mu\nu}^{(r)} - Q_2H_{\mu\nu}^{(r-1)} + Q_3H_{\mu\nu}^{(r-2)} \tag{39}$$

and

$$\Psi^{(r+1)} = Q_1\Psi^{(r)} - Q_2\Psi^{(r-1)} + Q_3\Psi^{(r-2)}. \tag{40}$$

This recurrence equation, in conjunction with eqns (35)–(37), enables $\Psi^{(3)}$, $\Psi^{(4)}$, ... to be obtained sequentially if required.

Note the remarkably simple result for $\Psi^{(0)}$ in eqn (35), as a result of which eqn (29) becomes

$$\xi = \frac{h^2}{12}\Psi^{(1)} - \frac{h^4}{120}\Psi^{(2)} + \frac{17h^6}{20{,}160}\Psi^{(3)} - \cdots. \tag{41}$$

It is apparent from this equation that ξ is of $O(h^2)$ and a first approximation for ξ can be obtained from the first term of eqn (41) alone, with $\Psi^{(1)}$ calculated from eqn (36) with $\xi = 0$. Thus

$$\xi = \frac{h^2}{12}[\bar{c}_{11}\alpha^4 + 2(\bar{c}_{12} + 2c_{66})\alpha^2\beta^2 + \bar{c}_{22}\beta^4] + O(h^4). \tag{42}$$

It will be seen later than this agrees with the result of classical thin plate theory for an orthotropic material.

A second approximation for ξ can now be obtained from the first two terms of eqn (41), using the first approximation (42) to adjust the expression for $\Psi^{(1)}$ in eqn (36), but with $\Psi^{(2)}$ calculated from eqn (37) with $\xi = 0$. After some algebraic manipulation this second approximation can be written as

$$\begin{aligned}\xi = {} & \frac{h^2}{12}\left[1 - \frac{h^2}{12}(\alpha^2 + \beta^2)\right](\alpha^2\eta_{11} + \beta^2\eta_{22}) \\ & - \frac{h^2}{120}\left[\left(\frac{1}{c_{55}} - \frac{c_{13}}{3\bar{c}_{11}c_{33}}\right)\alpha^2\eta_{11}^2 + \left(\frac{1}{c_{44}} - \frac{c_{23}}{3\bar{c}_{22}c_{33}}\right)\beta^2\eta_{22}^2\right. \\ & \left. + \frac{\alpha^2\beta^2}{3c_{33}}\left(\frac{c_{23}\alpha^2}{\bar{c}_{22}} + \frac{c_{13}\beta^2}{\bar{c}_{11}}\right)\{(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\}\right] + O(h^6)\end{aligned} \tag{43}$$

where, to save space, we have introduced the quantities

$$\eta_{11} = \bar{c}_{11}\alpha^2 + (\bar{c}_{12} + 2c_{66})\beta^2$$

and

$$\eta_{22} = \bar{c}_{22}\beta^2 + (\bar{c}_{12} + 2c_{66})\alpha^2. \tag{44}$$

In principle this process for obtaining approximations of progressively higher order can be continued indefinitely, but we shall not take it any further in the case of an orthotropic plate, partly because of the algebraic complexity and partly because eqn (43) is sufficient for our purposes. However, the equations for an isotropic plate are algebraically much simpler and in that case we have taken the process one stage further and obtained a third approximation for ξ. The result can be expressed very concisely if we introduce two dimensionless parameters ϕ and Δ, given by

$$\phi = \frac{(\alpha^2\sigma_x^0 + \beta^2\sigma_y^0 + \rho\omega^2)}{G(\alpha^2 + \beta^2)} \tag{45}$$

and

$$\Delta = (\alpha^2 + \beta^2)h^2/12. \tag{46}$$

In terms of these parameters, the third approximation is

$$\phi = \frac{2\Delta}{1-\nu}[1 - \mu_1\Delta + \mu_2\Delta^2] + O(\Delta^4) \tag{47}$$

where

$$\mu_1 = \frac{17-7\nu}{5(1-\nu)} \tag{48}$$

and

$$\mu_2 = \frac{62}{35} + \frac{(62-42\nu)}{5(1-\nu)^2}. \tag{49}$$

Although eqn (47) has been derived by reduction of the equations for an orthotropic plate, it can be derived in an alternative and more direct way if isotropy is assumed *ab initio*, thereby providing a welcome check on the correctness of the algebra in the orthotropic case. An outline of this alternative derivation is given in Appendix B.

Equations (43) and (47) provide a yardstick against which the accuracy of higher order plate theories can be measured. In particular we shall use them in Section 3.2 to investigate the accuracy of Mindlin's plate theory.

2.4. *The static loading problem and its solution*

We now turn to the problem of a simply supported rectangular plate, with sides a and b and with zero in-plane loads (i.e. $\sigma_x^0 = \sigma_y^0 = 0$), subjected to a static lateral load in the z-direction of magnitude p per unit area given by

$$p = P \sin \alpha x \sin \beta y$$

where α and β are such that $\alpha a/\pi$ and $\beta b/\pi$ are integers. We shall obtain a solution in the form of eqns (11) with $\omega = 0$.

In classical plate theory, and indeed in many higher order plate theories, it is immaterial whether the lateral loading is applied to just one surface, or shared between the two surfaces $z = \pm\frac{1}{2}h$. The reason for this is, of course, that w is assumed to be independent of z. In the three-dimensional theory, however, it is necessary to specify precisely how the load is applied. We shall assume that it is shared equally between the two surfaces, as a pressure $\frac{1}{2}p$ on one surface and a tension $\frac{1}{2}p$ on the other, as shown in Fig. 1(a). This results in an enormous simplification of the analysis, because the displacements are now antisymmetrical with respect to the middle plane $z = 0$. The boundary conditions to be satisfied are

$$\sigma_z = \pm\tfrac{1}{2}p, \qquad \tau_{xz} = \tau_{yz} = 0 \qquad \text{at } z = \pm\tfrac{1}{2}h. \tag{50}$$

It should be noted that if the load is applied as a pressure p on the surface $z = -\frac{1}{2}h$ only, the exact solution could be obtained by superposition of the solutions for the antisymmetric and symmetric loadings of Figs 1(a) and (b). Clearly the middle plane deflection in Fig. 1(b) is zero, so the middle plane deflection is due entirely to the antisymmetric load component of Fig. 1(a). On the other hand, if the lateral deflection of the loaded surface is required there is a small error incurred in ignoring the change in the half-thickness in Fig. 1(b). If we assume that $\sigma_z = -\frac{1}{2}p$ throughout the thickness and that the strains ε_x and ε_y are very small compared with ε_z in Fig. 1(b), it is easily seen that the amplitude of the change in the half-thickness, $\frac{1}{2}\delta h$, is approximately equal to $Ph/4c_{33}$. Now it will be found later that the amplitude of the lateral deflection of the plate due to the

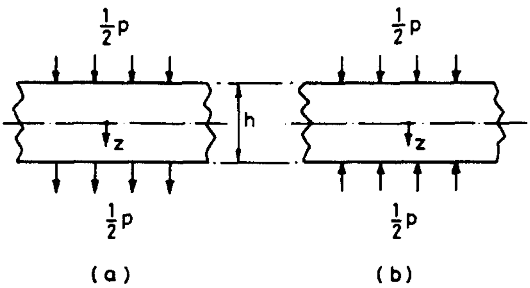

Fig. 1. (a) An antisymmetrical and (b) a symmetrical lateral loading.

lateral load p according to classical plate theory is W_{cl} where

$$W_{cl} = (12P/h^3)[\bar{c}_{11}\alpha^4 + 2(\bar{c}_{12} + 2c_{66})\alpha^2\beta^2 + \bar{c}_{22}\beta^4]^{-1}. \tag{51}$$

Hence

$$\tfrac{1}{2}\delta h/W_{cl} = (h^4/48c_{33})[\bar{c}_{11}\alpha^4 + 2(\bar{c}_{12} + 2c_{66})\alpha^2\beta^2 + \bar{c}_{22}\beta^4] \tag{52}$$

and we see that the change in half-thickness due to the symmetrical loading of Fig. 1(b) is of the order $(h^4/\lambda^4)W_{cl}$, where λ is a length typifying a half-wavelength of the mode. But we shall also find that the correction to the classical deflection due to transverse shear deformation is of order $(h^2/\lambda^2)W_{cl}$, and to the accuracy that we shall be concerned with it is therefore legitimate to replace the loading by that of Fig. 1(a). [An exception to this might, however, occur if c_{33} is small compared with $\bar{c}_{11}$, $\bar{c}_{22}$ or $(\bar{c}_{12} + 2c_{66})$.]

The general solution for the displacements due to the antisymmetric loading is given by eqns (20) with $K_i' = 0$. The boundary conditions (50) lead to three simultaneous equations for the K_i constants which are identical with (25) except that the zero on the right-hand side of the first equation is replaced by P/h. The solution of these equations can conveniently be written as

$$\theta K_k \cosh \tfrac{1}{2}hq_k = (P/h)[\alpha(f_i q_i^2 - f_j q_j^2) + \beta(e_j q_j^2 - e_i q_i^2) + (e_i f_j - e_j f_i)q_i^2 q_j^2] \tag{53}$$

where $(i, j, k) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$, and θ is the determinant of the matrix in eqn (25). But this determinant is given by the expression on the left-hand side of eqn (29), and we note also that $\xi = 0$, since σ_x^0, σ_y^0 and ω are all zero. Hence, using eqns (29) and (34)–(37), we have

$$\begin{aligned}\frac{\theta}{\kappa} = &-\frac{h^2}{12}\left[1 - \frac{h^2}{10}\left\{\left(\frac{\bar{c}_{11}}{c_{55}} - \frac{2c_{13}}{c_{33}}\right)\alpha^2 + \left(\frac{\bar{c}_{22}}{c_{44}} - \frac{2c_{23}}{c_{33}}\right)\beta^2\right\}\right](\alpha^2\eta_{11} + \beta^2\eta_{22}) \\ &+ \frac{h^4}{120}\left(\frac{\alpha^2}{c_{44}} + \frac{\beta^2}{c_{55}}\right)\alpha^2\beta^2[(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}] + O(h^6)\end{aligned} \tag{54}$$

where η_{11} and η_{22} are defined by eqns (44) and κ by eqn (33).

Consider now the lateral deflection of each of the two surfaces $z = \pm\tfrac{1}{2}h$, which we shall denote by $W_s \sin\alpha x \sin\beta y$. From the third of eqns (20) with $K_i' = 0$ we find that

$$W_s = \sum_{i=1}^{3} K_i \cosh \tfrac{1}{2}hq_i.$$

Hence, from (53)

$$\theta W_s = (P/h)\sum{}^*[\alpha(f_i q_i^2 - f_j q_j^2) + \beta(e_j q_j^2 - e_i q_i^2) + (e_i f_j - e_j f_i) q_i^2 q_j^2]$$

where the meaning of the summation $\sum^*$ is the same as in eqns (18) and (19). This equation can be written more concisely as

$$\theta W_s = (P/h)[-\alpha H_{31}^{(-1)} - H_{32}^{(-1)} + H_{33}^{(-1)}] \tag{55}$$

where the $H_{\mu\nu}^{(r)}$ quantities are as defined in eqns (32). But from eqn (39) we have

$$H_{\mu\nu}^{(-1)} = [H_{\mu\nu}^{(2)} - Q_1 H_{\mu\nu}^{(1)} + Q_2 H_{\mu\nu}^{(0)}]/Q_3$$

and hence, using Table 1, we find that

$$H_{31}^{(-1)} = H_{32}^{(-1)} = 0 \quad \text{and} \quad H_{33}^{(-1)} = -\kappa.$$

Equation (55) therefore becomes

$$W_s = -\frac{\kappa P}{h\theta}. \tag{56}$$

Finally, on substituting eqn (54) and using the binomial theorem we find that

$$\frac{W_s}{W_{cl}} = 1 + \frac{h^2}{10}\left[\left(\frac{\bar{c}_{11}}{c_{55}} - \frac{2c_{13}}{c_{33}}\right)\alpha^2 + \left(\frac{\bar{c}_{22}}{c_{44}} - \frac{2c_{23}}{c_{33}}\right)\beta^2 + \alpha^2\beta^2\left(\frac{\alpha^2}{c_{44}} + \frac{\beta^2}{c_{55}}\right)\frac{\{(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\right] + O(h^4) \tag{57}$$

where W_{cl} is the deflection according to classical plate theory, defined in eqn (51).

Let the lateral deflection of the middle plane be denoted by $W_0 \sin\alpha x \sin\beta y$. From the third of eqns (20) with $K_i' = 0$ we have

$$\begin{aligned} W_0 = \sum_{i=1}^{3} K_i &= \sum_{i=1}^{3} \operatorname{sech}\tfrac{1}{2}hq_i\{K_i \cosh\tfrac{1}{2}hq_i\} \\ &= \sum_{i=1}^{3}\left(1 - \frac{h^2}{8}q_i^2 + \ldots\right)K_i \cosh\tfrac{1}{2}hq_i \\ &= W_s - \frac{h^2}{8}\sum_{i=1}^{3} q_i^2 K_i \cosh\tfrac{1}{2}hq_i + \ldots. \end{aligned}$$

Hence, from eqn (53)

$$W_0 = W_s - \frac{Ph}{8\theta}\sum{}^*[\alpha q_k^2(f_i q_i^2 - f_j q_j^2) + \beta q_k^2(e_j q_j^2 - e_i q_i^2) + Q_3(e_i f_j - e_j f_i)] + \ldots.$$

After some cyclic modification of the suffixes i, j, k this becomes

$$W_0 = W_s - \frac{Ph}{8\theta}\sum{}^*[\alpha q_i^2 q_j^2(f_j - f_i) + \beta q_i^2 q_j^2(e_i - e_j) + Q_3(e_i f_j - e_j f_i)] + \ldots$$

which is more concisely written as

$$W_0 = W_s - \frac{Ph}{8\theta}[\alpha H_{31}^{(0)} + H_{32}^{(0)} + H_{33}^{(0)}] + \dots$$

Hence, using Table 1, and eqns (13) and (56), we obtain

$$W_0 = W_s\left[1 + \frac{h^2(c_{13}\alpha^2 + c_{23}\beta^2)}{8c_{33}} + O(h^4)\right]. \tag{58}$$

Finally, on substituting eqn (57) for W_s, we find that

$$\frac{W_0}{W_{cl}} = 1 + \frac{h^2}{10}\left[\left(\frac{\bar{c}_{11}}{c_{55}} - \frac{3c_{13}}{4c_{33}}\right)\alpha^2 + \left(\frac{\bar{c}_{22}}{c_{44}} - \frac{3c_{23}}{4c_{33}}\right)\beta^2 + \alpha^2\beta^2\left(\frac{\alpha^2}{c_{44}} + \frac{\beta^2}{c_{55}}\right)\frac{\{(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\right] + O(h^4). \tag{59}$$

In the special case of an isotropic plate, we have

$$\bar{c}_{11} = \bar{c}_{22} = (\bar{c}_{12} + 2c_{66}) = 2G/(1 - \nu). \tag{60}$$

Equations (57)–(59) are then much simpler

$$\frac{W_s}{W_{cl}} = 1 + (12/5)\Delta + O(\Delta^2) \tag{61}$$

$$\frac{W_0}{W_s} = 1 + \frac{3\nu}{2(1 - \nu)}\Delta + O(\Delta^2) \tag{62}$$

$$\frac{W_0}{W_{cl}} = 1 + \frac{3(8 - 3\nu)}{10(1 - \nu)}\Delta + O(\Delta^2) \tag{63}$$

where Δ is defined in eqn (46).

The difference between W_0 and W_s is not a trivial one. It is primarily due to the strains ε_z arising from the Poisson's ratio effect of the bending stresses σ_x and σ_y. This is easily demonstrated, as follows. According to classical plate theory the bending strains ε_x and ε_y are $z\alpha^2 W_{cl}$ and $z\beta^2 W_{cl}$ times $(\sin\alpha x \sin\beta y)$, respectively. Hence from eqns (4), assuming that σ_z is negligibly small, we find that

$$\varepsilon_z = -z(c_{13}\alpha^2 + c_{23}\beta^2)(W_{cl}/c_{33})\sin\alpha x \sin\beta y$$

and on integrating this with respect to z between 0 and $\frac{1}{2}h$ we obtain

$$W_0 - W_s = \frac{h^2(c_{13}\alpha^2 + c_{23}\beta^2)}{8c_{33}} W_{cl}$$

which agrees with eqn (58) to first order.

The magnitude of the difference is most easily appreciated if the plate is isotropic, in which case eqns (61)–(63) show that, to first order

$$\frac{W_0 - W_s}{W_s - W_{cl}} = \frac{5\nu}{8(1 - \nu)}.$$

This is equal to 0.3125 if $\nu = 1/3$ say, so that the change in the half-thickness is by no means negligible compared to the deflection due to shear. Now many higher order plate theories assume that w is constant through the thickness, and if so the question immediately arises as to whether one should compare it with W_s or W_0, or maybe some average value between the two. There appear to be at least two logical reasons for choosing W_s for this purpose. First the work done by the lateral pressure p is directly associated with W_s, so that if a higher order plate theory agrees with W_s it should correctly predict the total strain energy in the plate. Second, the middle plane deflection W_0 has no obvious physical significance and any experimental verification would inevitably involve measurement of W_s. An additional, though perhaps minor, advantage of W_s is that the first-order correction to W_{cl} in eqn (61) is independent of ν.

To conclude this section we consider the bending strains ε_x and ε_y at the two surfaces, which are of the form

$$\{\varepsilon_x; \varepsilon_y\} = \pm\{\hat{\varepsilon}_x; \hat{\varepsilon}_y\} \sin\alpha x \sin\beta y \quad \text{at } z = \pm\tfrac{1}{2}h. \tag{64}$$

It can be shown that

$$\hat{\varepsilon}_x = (P/2\theta)[\alpha^2 H_{11}^{(0)} - (h^2/12)\{\alpha^2 H_{11}^{(1)} + \alpha\beta H_{12}^{(1)} + \alpha H_{13}^{(1)}\} + O(h^4)]$$
$$\hat{\varepsilon}_y = (P/2\theta)[\beta^2 H_{22}^{(0)} - (h^2/12)\{\alpha\beta H_{21}^{(1)} + \beta^2 H_{22}^{(1)} + \beta H_{23}^{(1)}\} + O(h^4)].$$

On using Table 1, eqns (13) with $\xi = 0$, and eqns (51) and (54) we finally obtain the expressions

$$\begin{aligned}\hat{\varepsilon}_x &= \tfrac{1}{2}hW_{cl}\alpha^2\left[1 + \frac{h^2}{10}\left\{\frac{\beta^2\eta_{22}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\left(\frac{\eta_{22}}{c_{44}} - \frac{\eta_{11}}{c_{55}}\right) + \frac{\alpha^2}{6}\left(\frac{\bar{c}_{11}}{c_{55}} - \frac{7c_{13}}{c_{33}}\right)\right.\right.\\ &\quad\left.\left. + \frac{\beta^2}{6}\left(\frac{\bar{c}_{12} + 2c_{66}}{c_{55}} - \frac{7c_{23}}{c_{33}}\right)\right\} + O(h^4)\right]\\ \hat{\varepsilon}_y &= \tfrac{1}{2}hW_{cl}\beta^2\left[1 + \frac{h^2}{10}\left\{\frac{\alpha^2\eta_{11}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\left(\frac{\eta_{11}}{c_{55}} - \frac{\eta_{22}}{c_{44}}\right) + \frac{\beta^2}{6}\left(\frac{\bar{c}_{22}}{c_{44}} - \frac{7c_{23}}{c_{33}}\right)\right.\right.\\ &\quad\left.\left. + \frac{\alpha^2}{6}\left(\frac{\bar{c}_{12} + 2c_{66}}{c_{44}} - \frac{7c_{13}}{c_{33}}\right)\right\} + O(h^4)\right]\end{aligned} \tag{65}$$

where η_{11} and η_{22} are defined by eqns (44).

If the plate is isotropic the equations are much simpler and it is not difficult to obtain higher order approximations by assuming isotropy from the outset. The result for the fourth approximation is as follows

$$\frac{\hat{\varepsilon}_x}{\alpha^2} = \frac{\hat{\varepsilon}_y}{\beta^2} = \tfrac{1}{2}hW_{cl}\left[1 + \frac{(2 - 7\nu)}{5(1 - \nu)}\Delta + \frac{33}{175}\Delta^2 - \frac{34}{875}\Delta^3 + O(\Delta^4)\right]. \tag{66}$$

3. SOLUTIONS USING MINDLIN THEORY AND COMPARISONS WITH THE THREE-DIMENSIONAL ELASTICITY SOLUTIONS

In this section we shall solve the eigenvalue problem and the static loading problem using Mindlin's plate theory[2]. We shall compare the results with the solutions obtained in Sections 2.3 and 2.4, enabling us to draw certain conclusions about the Mindlin theory, and especially about the so-called "effective shear rigidities", that appear to be new.

In order to make a proper comparison with the three-dimensional theory, we need a version of Mindlin's plate equations which, in addition to transverse shear deformations, also incorporates:

(a) the general orthotropic constitutive equations, eqns (4),
(b) the effects of rotational inertia, and
(c) the complete quadratic expressions (3) for the second-order strains ε_x' and ε_y' when calculating the change of potential energy of the static, in-plane, compressive stresses σ_x^0 and σ_y^0.

Despite the extensive literature on Mindlin's plate theory the author has been unable to find a version in the published literature that satisfies all these requirements, and for that reason we list the relevant equations in the next section.

3.1. *The equations of Mindlin's plate theory for an orthotropic material*

The central assumptions of Mindlin's theory are that the lateral displacement w is independent of z, and that points on a normal to the middle plane before bending remain on a straight line, though not a normal to the middle surface, after bending. Thus the assumed displacement field is

$$u = -z\Psi_x(x, y, t), \qquad v = -z\Psi_y(x, y, t), \qquad w = w(x, y, t) \tag{67}$$

where Ψ_x and Ψ_y are rotations of the normal.

In relating the stresses σ_x and σ_y to the strains it is also assumed that the plate is in a state of plane stress, so that $\sigma_z = 0$. This enables ε_z to be related to ε_x and ε_y and then eliminated from the equations for σ_x and σ_y in eqns (4). The result is

$$\sigma_x = \bar{c}_{11}\varepsilon_x + \bar{c}_{12}\varepsilon_y, \qquad \sigma_y = \bar{c}_{12}\varepsilon_x + \bar{c}_{22}\varepsilon_y \tag{68}$$

where $\bar{c}_{11}$, $\bar{c}_{12}$ and $\bar{c}_{22}$ are defined by eqns (38). Also eqns (67) imply that the strains γ_{xz} and γ_{yz} are independent of z; to allow for the fact that this is not strictly correct the actual shear moduli c_{44} and c_{55} are replaced by empirical effective moduli c_{44}^* and c_{55}^*. Hence the shear stress–strain equations become

$$\tau_{yz} = c_{44}^*\gamma_{yz}, \qquad \tau_{xz} = c_{55}^*\gamma_{xz}, \qquad \tau_{xy} = c_{66}\gamma_{xy}. \tag{69}$$

The derivation of the governing differential equations and natural boundary conditions from Hamilton's principle now proceeds along standard lines, with the following results.

The stress resultants in the plate are given by

$$\begin{aligned}
M_x &= -\frac{h^3}{12}\left\{\bar{c}_{11}\frac{\partial \Psi_x}{\partial x} + \bar{c}_{12}\frac{\partial \Psi_y}{\partial y}\right\} \\
M_y &= -\frac{h^3}{12}\left\{\bar{c}_{12}\frac{\partial \Psi_x}{\partial x} + \bar{c}_{22}\frac{\partial \Psi_y}{\partial y}\right\} \\
M_{xy} &= -\frac{h^3}{12}c_{66}\left\{\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right\} \\
Q_x &= hc_{55}^*\left\{\frac{\partial w}{\partial x} - \Psi_x\right\} \\
Q_y &= hc_{44}^*\left\{\frac{\partial w}{\partial y} - \Psi_y\right\}.
\end{aligned} \tag{70}$$

The equations of motion are

$$-\frac{\partial Q_x}{\partial x}-\frac{\partial Q_y}{\partial y}+h\mathscr{L}(w)=p$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+\frac{h^3}{12}\mathscr{L}(\Psi_x)=0 \tag{71}$$

$$\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y+\frac{h^3}{12}\mathscr{L}(\Psi_y)=0$$

where $p(x, y)$ is the transverse load per unit area of plate, and $\mathscr{L}$ is the operator defined by eqn (2).

Substituting eqns (70) into eqns (71) gives the following three differential equations for w, Ψ_x and Ψ_y

$$-c_{55}^* h\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial \Psi_x}{\partial x}\right)-c_{44}^* h\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial \Psi_y}{\partial y}\right)+h\mathscr{L}(w)=p$$

$$\frac{-h^3}{12}\left[\bar{c}_{11}\frac{\partial^2\Psi_x}{\partial x^2}+c_{66}\frac{\partial^2\Psi_x}{\partial y^2}+(\bar{c}_{12}+c_{66})\frac{\partial^2\Psi_y}{\partial x\partial y}\right]-hc_{55}^*\left(\frac{\partial w}{\partial x}-\Psi_x\right)+\frac{h^3}{12}\mathscr{L}(\Psi_x)=0 \tag{72}$$

$$\frac{-h^3}{12}\left[\bar{c}_{22}\frac{\partial^2\Psi_y}{\partial y^2}+c_{66}\frac{\partial^2\Psi_y}{\partial x^2}+(\bar{c}_{12}+c_{66})\frac{\partial^2\Psi_x}{\partial x\partial y}\right]-hc_{44}^*\left(\frac{\partial w}{\partial y}-\Psi_y\right)+\frac{h^3}{12}\mathscr{L}(\Psi_y)=0.$$

The natural boundary conditions are as follows:

on an edge x = constant

$$\text{either } w \text{ is specified or } \left\{Q_x-h\sigma_x^0\frac{\partial w}{\partial x}\right\}=0,$$

$$\text{either } \Psi_x \text{ is specified or } \left\{M_x+\frac{h^3}{12}\sigma_x^0\frac{\partial \Psi_x}{\partial x}\right\}=0, \tag{73}$$

and

$$\text{either } \Psi_y \text{ is specified or } \left\{M_{xy}+\frac{h^3}{12}\sigma_x^0\frac{\partial \Psi_y}{\partial x}\right\}=0.$$

on an edge y = constant

$$\text{either } w \text{ is specified or } \left\{Q_y-h\sigma_y^0\frac{\partial w}{\partial y}\right\}=0,$$

$$\text{either } \Psi_y \text{ is specified or } \left\{M_y+\frac{h^3}{12}\sigma_y^0\frac{\partial \Psi_y}{\partial y}\right\}=0, \tag{74}$$

and

$$\text{either } \Psi_x \text{ is specified or } \left\{M_{xy}+\frac{h^3}{12}\sigma_y^0\frac{\partial \Psi_x}{\partial y}\right\}=0.$$

3.2. *Solution of the eigenvalue problem*

We shall now obtain a solution to the eigenvalue problem of Section 2.3 using Mindlin's theory. In Section 2.2 we defined the conditions of simple support in the three-dimensional theory as $w = 0$, $v = 0$, $\sigma_x = 0$ on the edges $x = 0, a$, and $w = 0$, $u = 0$, $\sigma_y = 0$ on the edges $y = 0, b$. For consistency, in Mindlin's theory we define them as $w = 0$, $\Psi_y = 0$, $M_x = 0$ on $x = 0, a$, and $w = 0$, $\Psi_x = 0$, $M_y = 0$ on $y = 0, b$. With the aid of the first two of eqns (70) we see that these are satisfied by a mode of the form

$$
\begin{aligned}
w &= W_m \sin \alpha x \sin \beta y \cos \omega t \\
\Psi_x &= R_x \cos \alpha x \sin \beta y \cos \omega t \\
\Psi_y &= R_y \sin \alpha x \cos \beta y \cos \omega t
\end{aligned}
\tag{75}
$$

where W_m, R_x and R_y are constants and α and β are such that $\alpha a/\pi$ and $\beta b/\pi$ are integers.

Substitution of eqns (75) into the differential equations, eqns (72), with $p = 0$ gives three homogeneous simultaneous equations for W_m, R_x and R_y and the requirement that the determinant of the matrix of coefficients should vanish leads to the following characteristic equation:

$$
\begin{aligned}
\xi = \frac{h^2}{12}\Bigg[&(\alpha^2\bar{c}_{11} + \beta^2 c_{66} - \xi)\left(\alpha^2 - \frac{\xi}{c^*_{55}}\right) \\
&+ (\beta^2\bar{c}_{22} + \alpha^2 c_{66} - \xi)\left(\beta^2 - \frac{\xi}{c^*_{44}}\right) + 2\alpha^2\beta^2(\bar{c}_{12} + c_{66})\Bigg] \\
&+ \frac{h^4}{144}\left[\frac{\alpha^2}{c^*_{44}} + \frac{\beta^2}{c^*_{55}} - \frac{\xi}{c^*_{44}c^*_{55}}\right][(\alpha^2\bar{c}_{11} + \beta^2 c_{66} - \xi)(\beta^2\bar{c}_{22} + \alpha^2 c_{66} - \xi) \\
&- \alpha^2\beta^2(\bar{c}_{12} + c_{66})^2].
\end{aligned}
\tag{76}
$$

If shear deformations are neglected, i.e. if $c^*_{44} \to \infty$ and $c^*_{55} \to \infty$, this reduces to

$$
\xi = (h^2/12)[\alpha^4\bar{c}_{11} + 2\alpha^2\beta^2(\bar{c}_{12} + 2c_{66}) + \beta^4\bar{c}_{22} - (\alpha^2 + \beta^2)\xi]. \tag{77}
$$

This includes the effects of rotational inertia, which explains the presence of the second-order term involving ξ on the right-hand side. If that term also is neglected we obtain the result of classical thin plate theory, in agreement with eqn (42) for the first approximation of the three-dimensional theory.

A second approximation can now be obtained by substituting the classical plate theory result for ξ into the right-hand side of eqn (76) and retaining terms of order h^2 and h^4. The result can be concisely expressed as

$$
\xi = \frac{h^2}{12}\left[1 - \frac{h^2}{12}(\alpha^2 + \beta^2)\right](\alpha^2\eta_{11} + \beta^2\eta_{22}) - \frac{h^4}{144}\left[\frac{\alpha^2\eta_{11}^2}{c^*_{55}} + \frac{\beta^2\eta_{22}^2}{c^*_{44}}\right] + O(h^6) \tag{78}
$$

where η_{11} and η_{22} are defined by eqns (44).

We can continue this process and obtain expressions of progressively higher order, but we have done so only in the case of an isotropic plate, for which the third approximation is

$$
\phi = \frac{2\Delta}{1 - \nu}[1 - \mu_1'\Delta + \mu_2'\Delta^2] + O(\Delta^4) \tag{79}
$$

Table 2. Comparison of the coefficients μ_2 and μ_2' in eqns (47) and (79) for the eigenvalue of an isotropic plate when $G^*/G = 5/(6-\nu)$ so that $\mu_1 = \mu_1'$

ν	μ_2 (eqn (47))	μ_2' (eqn (79))
0	13.971	13.960
0.25	19.727	19.604
0.3	21.527	21.380
0.3333	22.921	22.760
0.4	26.327	26.138
0.5	33.771	33.560

where

$$\mu_1' = 1 + \frac{2G}{G^*(1-\nu)} \tag{80}$$

$$\mu_2' = \mu_1'^2 + \mu_1' - 1 \tag{81}$$

and ϕ and Δ are defined by eqns (45) and (46).

Let us now compare these results with the corresponding ones of the three-dimensional theory of elasticity. It is convenient to start with the isotropic case and compare eqns (47) and (79). We see that the second approximations agree if $\mu_1 = \mu_1'$ and eqns (48) and (80) show that this requires the effective shear modulus G^* to be given by

$$G^*/G = 5/(6-\nu). \tag{82}$$

Mindlin[2] assumed that $G^*/G = \pi^2/12 = 0.822$, a value that he obtained by making the frequency of the first antisymmetrical mode of thickness-shear vibration (in which $v = w = 0$ everywhere and $u = U(z)\cos\omega t$) coincide with the exact one. It is implicit in Reissner's plate theory[1] on the other hand, that $G^*/G = 5/6 = 0.833$. It is, however, known from *numerical* investigations[4, 7] that both of these values lead to natural frequencies of lateral vibration that are slightly low, and both Srinivas *et al.*[4] and Dawe[8] observed that a value of about 0.88 gave the best agreement with exact values. They both assumed $\nu = 0.3$ in their numerical work but Dawe[8] conjectured that the optimum value of G^*/G would depend upon Poisson's ratio. This is now confirmed by eqn (82), which gives 0.877 if $\nu = 0.3$, in very close agreement with the value 0.88.

Having forced the second approximations in eqns (47) and (79) to agree by choosing G^*/G in accordance with eqn (82), it is of interest now to consider the coefficients μ_2 and μ_2'. They are compared in Table 2 and it will be seen that over the entire range $0 < \nu < 0.5$ they are in astonishingly close agreement. We conclude therefore that if G^* is chosen in accordance with eqn (80), not only does the second approximation obtained from Mindlin's theory agree precisely with that of the three-dimensional theory but also the third approximations are in almost exact agreement.

These conclusions are independent of the mode, i.e. of the relative values of α and β. In particular they remain valid if β, say, is zero. The mode is then cylindrical, and the plate can be thought of as a simply-supported beam of wide rectangular cross-section which is subjected to a longitudinal compressive stress σ_x^0 and undergoes either buckling or free lateral vibrations. Thus eqn (82) also gives the effective shear modulus for buckling or vibration of a Timoshenko beam of *wide* rectangular cross-section. Now in that case the material will behave essentially as if it were in a state of plane strain, with $\varepsilon_y = 0$, for which the relevant stress–strain equations are

$$\begin{aligned}\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\frac{\partial w}{\partial z}\right]\\ \sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\frac{\partial u}{\partial x}\right]\\ \tau_{xz} &= \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right].\end{aligned} \tag{83}$$

These equations are precisely the same as

$$\begin{aligned}\sigma_x &= \frac{E_0}{(1-\nu_0^2)}\left[\frac{\partial u}{\partial x} + \nu_0\frac{\partial w}{\partial z}\right]\\ \sigma_z &= \frac{E_0}{(1-\nu_0^2)}\left[\frac{\partial w}{\partial z} + \nu_0\frac{\partial u}{\partial x}\right]\\ \tau_{xz} &= \frac{E_0}{2(1+\nu_0)}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]\end{aligned} \tag{84}$$

if

$$E_0 = E/(1-\nu^2), \qquad \nu_0 = \nu/(1-\nu) \tag{85}$$

or conversely

$$E = E_0(1+2\nu_0)/(1+\nu_0)^2, \qquad \nu = \nu_0/(1+\nu_0). \tag{86}$$

But eqns (84) are the stress–strain relations for a state of generalized plane *stress* in an isotropic material with Young's modulus E_0 and Poisson's ratio ν_0. We can thence deduce that the effective shear modulus for buckling or vibration of a Timoshenko beam of *narrow* rectangular cross-section is given by eqn (82) with ν replaced by $\nu/(1+\nu)$, i.e.

$$G^*/G = 5(1+\nu)/(6+5\nu). \tag{87}$$

If $\nu = 0.3$ eqns (82) and (87) give $G^*/G = 0.877$ and 0.867, respectively, so there is not much difference between them in practice.

Let us now return to the orthotropic plate, and compare the second approximation of Mindlin's theory (eqn (78)) with that of the three-dimensional theory (eqn (43)). We see immediately that it is impossible in general to choose values for Mindlin's effective shear moduli c_{44}^* and c_{55}^* which are independent of the mode (i.e. of the ratio α/β) and which make eqns (43) and (78) agree. The reason for this is the presence of the term

$$\frac{\alpha^2\beta^2}{3c_{33}}\left(\frac{c_{23}\alpha^2}{\bar{c}_{22}} + \frac{c_{13}\beta^2}{\bar{c}_{11}}\right)\{(\bar{c}_{12}+2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\} \tag{88}$$

in eqn (43), and the absence of a matching term in eqn (78).

There are, however, two exceptional circumstances in which the term vanishes. The first is the very restrictive case of an orthotropic material with elastic constants satisfying the condition

$$\bar{c}_{11}\bar{c}_{22} = (\bar{c}_{12} + 2c_{66})^2. \tag{89}$$

This is satisfied by isotropic materials, for which $\bar{c}_{11} = \bar{c}_{22} = (\bar{c}_{12} + 2c_{66}) = 2G/(1 - \nu)$. The second exception is when β (or α) is zero; as we saw previously this represents the case of an orthotropic Timoshenko beam with a *wide* rectangular cross-section. In either of these two circumstances eqns (43) and (78) agree exactly if

$$\begin{aligned} \frac{1}{c_{44}^*} &= \frac{6}{5}\left[\frac{1}{c_{44}} - \frac{c_{23}}{3(c_{22}c_{33} - c_{23}^2)}\right] \\ \frac{1}{c_{55}^*} &= \frac{6}{5}\left[\frac{1}{c_{55}} - \frac{c_{13}}{3(c_{11}c_{33} - c_{13}^2)}\right] \end{aligned} \tag{90}$$

where we have used eqns (38) for $\bar{c}_{11}$ and $\bar{c}_{22}$. For an isotropic material eqns (90) give $c_{44}^* = c_{55}^* = 5G/(6 - \nu)$, in agreement with eqn (82).

It is possible to use an argument similar to that used in deriving eqn (87) to deduce a value for c_{55}^* appropriate to the buckling or free vibration of an orthotropic Timoshenko beam with a rectangular cross-section that is narrow in the y-direction. The result can be obtained by replacing c_{11}, c_{33} and c_{13} in the second of eqns (90) by $c_{11} - (c_{12}^2/c_{22})$, $c_{33} - (c_{23}^2/c_{22})$ and $c_{13} - (c_{12}c_{23}/c_{22})$, respectively.

3.3. *Solution of the static loading problem*

Consider now the problem of static loading described in Section 2.4. The solution by Mindlin's theory is obtained by substituting eqns (75) with $\omega = 0$ into the differential equations, eqns (72), with $\mathscr{L} = 0$ and $p = P\sin\alpha x \sin\beta y$. This gives three simultaneous equations for W_{m}, R_x and R_y and on solving for W_{m}, and expanding in powers of h^2, we obtain the following second approximation

$$\frac{W_{\mathrm{m}}}{W_{\mathrm{cl}}} = 1 + \frac{h^2}{12}\left[\frac{\bar{c}_{11}\alpha^2}{c_{55}^*} + \frac{\bar{c}_{22}\beta^2}{c_{44}^*} + \alpha^2\beta^2\left(\frac{\alpha^2}{c_{44}^*} + \frac{\beta^2}{c_{55}^*}\right)\frac{\{(\bar{c}_{12} + 2c_{66})^2 - \bar{c}_{11}\bar{c}_{22}\}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\right] + O(h^4). \tag{91}$$

The bending strains at the surfaces $z = \pm\frac{1}{2}h$ are given by eqn (64), and from eqns (67) we have

$$\hat{\varepsilon}_x = \tfrac{1}{2}h\alpha R_x, \qquad \hat{\varepsilon}_y = \tfrac{1}{2}h\beta R_y. \tag{92}$$

On solving for R_x and R_y, and again expanding in powers of h^2, we find that

$$\begin{aligned} \hat{\varepsilon}_x &= \tfrac{1}{2}hW_{\mathrm{cl}}\alpha^2\left[1 + \frac{h^2}{12}\frac{\beta^2\eta_{22}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\left(\frac{\eta_{22}}{c_{44}^*} - \frac{\eta_{11}}{c_{55}^*}\right) + O(h^4)\right] \\ \hat{\varepsilon}_y &= \tfrac{1}{2}hW_{\mathrm{cl}}\beta^2\left[1 + \frac{h^2}{12}\frac{\alpha^2\eta_{11}}{(\alpha^2\eta_{11} + \beta^2\eta_{22})}\left(\frac{\eta_{11}}{c_{55}^*} - \frac{\eta_{22}}{c_{44}^*}\right) + O(h^4)\right]. \end{aligned} \tag{93}$$

If the plate is isotropic the equations are greatly simplified and the solution can be shown to be

$$\frac{W_{\mathrm{m}}}{W_{\mathrm{cl}}} = 1 + \frac{2G}{(1 - \nu)G^*}\Delta \tag{94}$$

and

$$\hat{\varepsilon}_x/\alpha^2 = \hat{\varepsilon}_y/\beta^2 = \tfrac{1}{2}hW_{cl}. \tag{95}$$

Note that there is no error in eqns (94) and (95) other than the inherent one arising from the basic assumptions of Mindlin's theory.

Let us now compare this solution with that of the three-dimensional theory, and we start with the isotropic case. From eqns (94) and (61) we see that W_m agrees to second order with the surface deflection, W_s, if

$$\frac{G^*}{G} = \frac{5}{6(1-\nu)}. \tag{96}$$

As discussed previously, this would also ensure that Mindlin's theory gives the strain energy correct to second order.

[We could alternatively make W_m agree to second order with the middle plane deflection W_0 of eqn (63), by choosing $G^*/G = 20/(24 - 9\nu)$, but the author can see no good reason for choosing W_0 in preference to W_s.]

We see from eqn (95) that the bending strains of the isotropic Mindlin and classical theories are identical, and agree only to first order with those of the three-dimensional theory in eqns (66). Note, however, that the coefficient of Δ in eqn (66) is zero if $\nu = 2/7 = 0.286$, and the Mindlin and classical theories then give strains correct to second order.

Turning now to the orthotropic case, we see from eqns (91) and (57) that it is in general impossible to choose values for c_{44}^* and c_{55}^* which are both independent of α/β and make W_m and W_s agree to second order. There are, however, the same two exceptions as in the eigenvalue problem, namely the case of orthotropic materials with elastic constants satisfying eqn (89), which encompasses isotropic materials, and the case of cylindrical bending when β (or α) is zero. In both cases W_m and W_s agree to second order if

$$\frac{1}{c_{44}^*} = \frac{6}{5}\left[\frac{1}{c_{44}} - \frac{2c_{23}}{(c_{22}c_{33} - c_{23}^2)}\right]$$

and (97)

$$\frac{1}{c_{55}^*} = \frac{6}{5}\left[\frac{1}{c_{55}} - \frac{2c_{13}}{(c_{11}c_{33} - c_{13}^2)}\right].$$

These both reduce to eqn (96) in the case of an isotropic material.

As in the eigenvalue problem, if $\beta \to 0$ the problem becomes one of a simply supported beam of wide rectangular cross-section carrying a lateral pressure p that varies sinusoidally in the longitudinal x-direction. The Mindlin plate becomes a Timoshenko beam and eqn (96) or eqn (97) gives the appropriate effective shear modulus, depending upon whether it is isotropic or orthotropic. These equations are independent of α and since any distribution of lateral load can be considered as a series of sinusoidally distributed loads of the type considered here it follows that eqn (96) or eqn (97) gives the optimum value of the effective shear modulus for any smoothly varying load, including a uniform one.

The corresponding value for a narrow rectangular cross-section can be deduced in a manner exactly similar to that used in Section 3.2. In particular, for an isotropic beam, replacing ν by $\nu/(1 + \nu)$ in eqn (96) gives

$$G^*/G = 5(1 + \nu)/6. \tag{98}$$

A check on this value can be obtained using the known solution for a simply supported beam of span l and narrow rectangular cross-section carrying a uniform lateral pressure q, obtained by means of an Airy stress function in the form of a polynomial[9]. According to that solution the *surface* deflection at mid-span is

$$\frac{5}{32}\frac{ql^4}{Eh^3}\left[1+\frac{48}{25}\frac{h^2}{l^2}+O\left(\frac{h^4}{l^4}\right)\right].$$

Timoshenko beam theory, on the other hand, gives

$$\frac{5}{32}\frac{ql^4}{Eh^3}+\frac{ql^2}{8G^*h}=\frac{5}{32}\frac{ql^4}{Eh^3}\left[1+\frac{4}{5}\frac{E}{G^*}\frac{h^2}{l^2}\right]$$

which agrees with the stress function solution to second order if G^* is given by eqn (98).

REFERENCES

1. E. Reissner, The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.* **12**, A69 (1945).
2. R. D. Mindlin, Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *J. Appl. Mech.* **18**, 31 (1951).
3. S. R. Srinivas and A. K. Rao, Buckling of thick rectangular plates. *AIAA J.* **7**, 1645 (1969).
4. S. R. Srinivas, C. V. Joga Rao and A. K. Rao, An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates. *J. Sound Vib.* **12**, 187 (1970).
5. S. R. Srinivas and A. K. Rao, Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. *Int. J. Solids Structures* **6**, 1463 (1970).
6. S. Srinivas and A. K. Rao, Flexure of thick rectangular plates. *J. Appl. Mech.* **40**, 298 (1973).
7. N. J. Pagano, Exact solutions for rectangular bidirectional composites and sandwich plates. *J. Composite Materials* **4**, 20 (1970).
8. D. J. Dawe, Finite strip models for vibration of Mindlin plates. *J. Sound Vib.* **59**, 441 (1978).
9. S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 2nd edn, Chap. 3. McGraw-Hill, New York (1951).

APPENDIX A: THE SUMMATIONS $I_{m,n}$ AND $J_{m,n}$

This Appendix is concerned with relations between the quantities $I_{m,n}$ and $J_{m,n}$ and in particular we show how they can all be expressed as $J_{1,1}$ times a polynomial in Q_1, Q_2 and Q_3.

First consider $I_{m,n}$ defined by eqn (18). Without altering this definition we can modify the suffixes of the two terms within the summation, in the first by cycling forward so that $(i,j,k)\rightarrow(j,k,i)$, and in the second by cycling backward so that $(i,j,k)\rightarrow(k,i,j)$. Hence we find that

$$I_{m,n}=\sum{}^*[q_i^{2m}q_j^{2n}-q_j^{2m}q_i^{2n}] \tag{A1}$$

so that

$$I_{m,n}=-I_{n,m} \tag{A2}$$

which implies

$$I_{n,n}=0. \tag{A3}$$

With eqn (A2) in mind we may now concentrate on the case where $m>n$ and we see from eqns (A1) and (19) that

$$I_{m,n}=J_{n,m-n}\qquad \text{if } m>n. \tag{A4}$$

Equations (A2) and (A4) enable all summations of the $I_{m,n}$ type to be expressed in terms of summations of the $J_{m,n}$ type, which we now proceed to consider.

Note first that

$$J_{0,n}=J_{m,0}=0. \tag{A5}$$

Note also that

$$J_{1,2} = \sum^{*} q_i^2 q_j^2 (q_i^4 - q_j^4) = \sum^{*} q_i^2 q_j^2 (Q_1 - q_k^2)(q_i^2 - q_j^2) = Q_1 J_{1,1} - Q_3 J_{0,1}$$

whence, using eqn (A5), we obtain the identity

$$J_{1,2} = Q_1 J_{1,1}. \tag{A6}$$

By similar manipulations we can show that

$$J_{2,1} = Q_2 J_{1,1} \tag{A7}$$

and

$$J_{2,2} = (Q_1 Q_2 - Q_3) J_{1,1}. \tag{A8}$$

Moreover, the following recurrence formulae can be derived:

$$J_{m,n+1} = Q_1 J_{m,n} - Q_2 J_{m,n-1} + Q_3 J_{m,n-2} \qquad \text{for } n \geqq 2 \tag{A9}$$

and

$$J_{m+1,n} = Q_2 J_{m,n} - Q_1 Q_3 J_{m-1,n} + Q_3^2 J_{m-2,n} \quad \text{for } m \geqq 2. \tag{A10}$$

Repeated applications of eqns (A9) and (A10), making use of eqns (A6)–(A8), enable as many of the $J_{m,n}$ summations as are needed to be expressed in terms of $J_{1,1}$ times a polynomial in Q_1, Q_2 and Q_3.

APPENDIX B: DIRECT DERIVATION OF EQUATION (47)

In the special case of an isotropic material two of the roots q_i of the auxiliary equation of eqn (15) are equal. The roots are in fact given by

$$\begin{aligned} q_1^2 &= q_2^2 = (\alpha^2 + \beta^2)(1 - \phi) \\ q_3^2 &= (\alpha^2 + \beta^2)\left[1 - \frac{(1 - 2\nu)}{2(1 - \nu)}\phi\right] \end{aligned} \tag{B1}$$

where ϕ is defined by eqn (45).

The general solution for $U(z)$, $V(z)$ and $W(z)$ in which U and V are odd-valued and W an even-valued function can be shown to be

$$\begin{aligned} U &= \alpha(K_1 \sinh q_2 z + K_3 \sinh q_3 z) \\ V &= \beta(K_2 \sinh q_2 z + K_3 \sinh q_3 z) \\ W &= q_2^{-1}(\alpha^2 K_1 + \beta^2 K_2)\cosh q_2 z + q_3 K_3 \cosh q_3 z. \end{aligned} \tag{B2}$$

Using this solution, the condition that the surfaces $z = \pm\frac{1}{2}h$ are stress free leads to three homogeneous simultaneous equations for the constants of integration K_1, K_2 and K_3. The requirement that the determinant of the matrix of coefficients must vanish gives the characteristic equation, which can be arranged in the following form:

$$\left[1 - \frac{q_2^2}{\alpha^2 + \beta^2}\right]^2 = \frac{4q_2^2}{\alpha^2 + \beta^2}\left[\frac{\bar{q}_3 \tanh \bar{q}_2}{\bar{q}_2 \tanh \bar{q}_3} - 1\right] \tag{B3}$$

where

$$\bar{q}_2 = \tfrac{1}{2} q_2 h, \qquad \bar{q}_3 = \tfrac{1}{2} q_3 h.$$

The left-hand side of this equation is simply equal to ϕ^2 and after expanding the term in brackets on the right-hand side in a power series, using eqns (B1), and dividing throughout by ϕ, we obtain

$$\phi = \frac{2\Delta}{1 - \nu}\left[1 - \left(\phi + \frac{7}{5}\Delta\right) + \left\{\frac{62}{35}\Delta^2 + \frac{27 - 28\nu}{10(1 - \nu)}\phi\Delta\right\} + O(\Delta^3)\right]. \tag{B4}$$

The grouping of the terms on the right-hand side of eqn (B4) reflects the fact that ϕ is of order Δ. Now collecting together the terms involving ϕ we obtain

$$\phi\left[1 + \frac{2\Delta}{1-\nu} - \frac{(27-28\nu)}{5(1-\nu)^2}\Delta^2 + O(\Delta^3)\right] = \frac{2\Delta}{1-\nu}\left[1 - \frac{7}{5}\Delta + \frac{62}{35}\Delta^2 + O(\Delta^3)\right]. \tag{B5}$$

Finally, after inverting the series on the left-hand side, we obtain eqn (47).